(2) Q No. → Examine the Convergence of the series

$$\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \frac{(\log 4)^2}{4^2} + \cdots + \frac{(\log n)^2}{n^2} + \cdots \text{ to } \infty.$$

Ans. → Let the general term of the given series be denoted by $f(n)$.

$$\text{Here, } f(n) = \frac{(\log n)^2}{n^2}$$

we know that $f(n)$ is +ve decreasing function of n, because n increases at a rate faster than $\log n$.

Let us suppose that a +ve integer >1.

$$a^n f(a^n) = \frac{a^n (\log a^n)^2}{(a^n)^2} = \frac{(n \log a)^2}{a^n}$$

$$= \frac{(\log a)^2}{a^n} \cdot n^2$$

Let the given term be denoted by u_n

$$u_n = \frac{(\log a)^2}{a^n} \cdot n^2$$

Replacing n by $n+1$, we have

$$u_{n+1} = \frac{(\log a)^2}{a^{n+1}} \cdot \frac{(n+1)^2}{1}$$

$$\boxed{\therefore \frac{u_{n+1}}{u_n} = \frac{(\log a)^2}{a^{n+1}} \cdot (n+1)^2 \times \frac{\log a^2}{a^n} \cdot n^2}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{\cancel{(\log a)^2}}{\cancel{a^{n+1}}\,a} \cdot \frac{(n+1)^2}{1} \times \frac{\cancel{a^n}}{\cancel{(\log a)^2}\,n^2}$$

$$= \frac{(n+1)^2}{n^2} \cdot \frac{1}{a}$$

$$= \frac{\cancel{n^2}(1+1/n)^2}{\cancel{n^2}} \cdot \frac{1}{a}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(1+1/n)^2}{} \cdot \frac{1}{a} = \left(1+\frac{1}{\infty}\right)^2 \cdot \frac{1}{a}$$

$$= \frac{1}{a}$$

$\therefore$ From D' Alembert's ratio test, Σu_n that is, $\Sigma a^n f(a^n)$ is Convergent and the given series is Convergent, when $\frac{1}{a} < 1$, as $a > 1$

Hence from Cauchy's Condensation test the given series is Convergent.

Q.No.→ Test for Convergence the following series for $p \in R$:-

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \cdots \frac{1}{(\log n)^p} + \cdots \text{ to } \infty.$$

Ans.→ Let the general term of the given series be denoted by $f(n)$

$$\text{Here, } f(n) = \frac{1}{(\log n)^p}$$

Let us suppose that a +ve integer >1

$$a^n f(a^n) = a^n \frac{1}{(\log a^n)^p} = \frac{a^n}{(n \log a)^p}$$

$$= \frac{a^n}{n^p (\log a)^p}$$

Let the given term be denoted by u_n

$$u_n = \frac{a^n}{n^p (\log a)^p}$$

Replacing n by $n+1$, we have

$$u_{n+1} = \frac{a^{n+1}}{(n+1)^p (\log a)^p}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{a\,\cancel{a^{n}}}{(n+1)^p \cancel{(\log a)^p}} \times \frac{n^p \cancel{(\log a)^p}}{\cancel{a^n}}$$

$$= \frac{n^p}{(n+1)^p} a = \frac{\cancel{n^p}}{\cancel{n^p}(1+\frac{1}{n})^p} \cdot a$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^p} \cdot a = \frac{1}{(1+\frac{1}{\infty})^p} \cdot a = \frac{1}{1+0} \cdot a = a$$

Hence, from D' Alembert's ratio test, Σu_n that is, $\Sigma a^n f(a^n)$ is convergent.

$\therefore$ from Cauchy's condensation test $\Sigma f(n)$ is also divergent, when $p>0$.

⑧ QNo→ Test for convergence the series

$$\sum_{n=2}^{\infty} \frac{\log n}{n^p}.$$

Ans.→ Let the general term of the given series be denoted by $f(n)$

$$\text{Then, } f(n) = \frac{\log n}{n^p}$$

Let $p \leq 0$ and $p = -m$, where $m \geq 0$

$$\therefore f(n) = \frac{\log n}{n^{-m}} = n^m \log n$$

$$\text{or, } \lim_{n \to \infty} f(n) = \infty^m \log \infty = \infty.$$

Hence, the given series is divergent when $p \leq 0$

Let $p > 0$, Here $f(n)$ is a +ve decreasing function, because n^p increases at a rate faster than $\log n$

Let us suppose that a +ve integer >1

$$\therefore a^n f(a^n) = a^n \cdot \frac{\log a^n}{(a^n)^p} = \frac{n \log a}{a^{n(p-1)}}$$

Let the given term be denoted by u_n

$$u_n = \frac{n \log a}{a^{n(p-1)}}$$

Replacing n by $(n+1)$, we have

$$u_{n+1} = \frac{(n+1)\log a}{a^{n+1}(p-1)}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)\log a}{a^{n+1}(p-1)} \times \frac{a^{n}(p-1)}{n \log a}$$

$$= \frac{(n+1)}{n} \cdot \frac{1}{a^{p-1}}$$

$$= \frac{\cancel{n}(1+1/n)}{\cancel{n}} \cdot \frac{1}{a^{p-1}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} (1+1/n) \cdot \frac{1}{a^{p-1}}$$

$$= \frac{1}{a^{p-1}}$$

Hence, from D' Alembert's ratio test, Σu_n that is, $\Sigma a^n f(a^n)$ is convergent, or divergent.

$\therefore$ from Cauchy's condensation test, the given series $\Sigma f(n)$ is ~~also~~ convergent, when $p>1$ and divergent, when $p \leq 1$.